HEISENBERG UNIQUENESS PAIRS FOR SOME ALGEBRAIC CURVES IN THE PLANE

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ABSTRACT. A Heisenberg uniqueness pair is a pair (Γ, Λ), where Γ is a curve and Λ is a set in $\mathbb{R}^2$ such that whenever a finite Borel measure μ having support on Γ which is absolutely continuous with respect to the arc length on Γ satisfies $\hat{\mu}|_\Lambda = 0$, then it is identically 0. In this article, we investigate the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and certain exponential curves. Further, we work out a characterization of the Heisenberg uniqueness pairs corresponding to four parallel lines. In the latter case, we observe a phenomenon of interlacing of three trigonometric polynomials.

1. INTRODUCTION

The concept of the Heisenberg uniqueness pair has been first introduced in an influential article by Hedenmalm and Montes-Rodrguez (see [7]). We would like to mention that Heisenberg uniqueness pair up to a certain extent is similar to an annihilating pair of Borel measurable sets of positive measure as described by Havin and Joricke [6]. Further, the notion of Heisenberg uniqueness pair has a sharp contrast to the known results about determining sets for measures by Sitaram et al. [3, 14], due to the fact that the determining set Λ for the function $\hat{\mu}$ has also been considered a thin set.

In addition, the question of determining the Heisenberg uniqueness pair for a class of finite measures has also a significant similarity with the celebrated result due to M. Benedicks (see[1]). That is, support of a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}$ cannot be of finite measure simultaneously. Later, various analogues of the Benedicks theorem have been investigated in different set ups, including the Heisenberg group and Euclidean motion groups (see [12, 15, 18]).

In particular, if Γ is compact, then $\hat{\mu}$ is a real analytic function having exponential growth and it can vanish on a very delicate set. Hence in this case, finding the Heisenberg uniqueness pairs becomes little easier. However, this question becomes immensely difficult when the measure is supported on a non-compact curve. Eventually, the Heisenberg uniqueness pair is a natural

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invariant to the theme of the well studies uncertainty principle for the Fourier transform.

In the article [7], Hedenmalm and Montes-Rodrguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein-Gordon equation. Further, a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines has been given by Hedenmalm and Montes-Rodrguez (see [7]).

Afterward, a considerable amount of work has been done pertaining to the Heisenberg uniqueness pair in the plane as well as in the higher dimensional Euclidean spaces.

Recently, N. Lev [10] and P. Sjolin [16] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle S^1. Further, F. J. Gonzalez Vieli [19] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions $J_{(n+2k-2)/2}$; $k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}.

Per Sjolin [17] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. Subsequently, D. Blasi Babot [2] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. However, an exact analogue for the finitely many parallel lines is still open.

In a major development, P. Jaming and K. Kellay [8] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$, via dynamical system approach.

Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure μ in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ is defined by

$$\hat{\mu}(\xi, \eta) = \int_\Gamma e^{-i\pi(x\cdot\xi + y\cdot\eta)} d\mu(x, y).$$

In the above context, the function $\hat{\mu}$ becomes a uniformly continuous bounded function on $\mathbb{R}^2$. Thus, we can analyze the pointwise vanishing nature of the function $\hat{\mu}$.

Definition 1.1. Let Λ be a set in $\mathbb{R}^2$. The pair (Γ, Λ) is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfying $\hat{\mu}|_\Lambda = 0$, implies $\mu = 0$.

Since the Fourier transform is invariant under translation and rotation, one can easily deduce the following invariance properties about the Heisenberg uniqueness pair.

(i) Let $u_o, v_o \in \mathbb{R}^2$. Then the pair (Γ, Λ) is a HUP if and only if the pair $(\Gamma + u_o, \Lambda + v_o)$ is a HUP.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transform whose adjoint is denoted by T^*. Then (Γ, Λ) is a HUP if and only if $(T^{-1}\Gamma, T^*\Lambda)$ is a HUP.

Now, we state first known results on the Heisenberg uniqueness pair due to Hedenmalm and Montes-Rodrguez [7]. After that, we briefly indicate the progress on this recent problem.

Theorem 1.2. [7] *Let $\Gamma = L_1 \cup L_2$, where L_j; $j = 1, 2$ are any two parallel straight lines and Λ a subset of $\mathbb{R}^2$ such that $\overline{\pi(\Lambda)} = \mathbb{R}$. Then (Γ, Λ) is a Heisenberg uniqueness pair if and only if the set*

(1.1) $$\widetilde{\Lambda} = \pi_1^a(\Lambda) \cup \left[\pi_1^b(\Lambda) \smallsetminus \pi_1^c(\Lambda)\right]$$

is dense in $\mathbb{R}$.

Here we avoid to mention the notations appeared in (1.1) as they are bit involved, however, we have written down the same notations as in the article [7]. Though, their main features can be perceived in Section 3.

Theorem 1.3. [7] *Let Γ be the hyperbola $x_1 x_2 = 1$ and $\Lambda_{\alpha,\beta}$ a lattice-cross defined by*

$$\Lambda_{\alpha,\beta} = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals. Then $(\Gamma, \Lambda_{\alpha,\beta})$ is a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1$.

For $\xi \in \Lambda$, define a function e_ξ on Γ by $e_\xi(x) = e^{i\pi x \cdot \xi}$. As a dual problem to Theorem 1.3, Hedenmalm and Montes-Rodriguez [7] have proved the following density result which in turn solve the one-dimensional Kein-Gordon equation.

Theorem 1.4. [7] *The pair (Γ, Λ) is a Heisenberg uniqueness pair if and only if the set $\{e_\xi : \xi \in \Lambda\}$ is a weak* dense subspace of $L^\infty(\Gamma)$.*

Remark 1.5. In particular, for Γ to be an algebraic curve, the question of Heisenberg uniqueness pair can be understood through a partial differential equation (PDE). That is, if Γ is the zero set of a polynomial P on $\mathbb{R}^2$, then $\hat{\mu}$ satisfies the PDE

$$P\left(\frac{\partial_1}{i\pi}, \frac{\partial_2}{i\pi}\right)\hat{\mu} = 0$$

with initial condition $\hat{\mu}|_\Lambda = 0$. This formulation may help potentially in determining the geometrical structure of the set $Z(\hat{\mu})$, the zero set of the function $\hat{\mu}$. If we consider Λ to be contained in $Z(\hat{\mu})$, then (Γ, Λ) is not a HUP. Hence the question of the HUP arises when Λ has located away from $Z(\hat{\mu})$.

In the case when μ is supported on a circle, the function $\hat{\mu}$ becomes real analytic and hence it could vanish at most on a very thin set. Thus, there are an enormous number of candidates for Λ such that (Γ, Λ) is a HUP. Some of the

Heisenberg uniqueness pairs corresponding to circle has been independently investigated by N. Lev and P. Sjolin. Following are their main results. For more details, we refer to [10, 16].

Theorem 1.6. [10, 16] *Let $\Gamma = S^1$ be the unit circle.*

(i) Let Λ be a circle of radius r. Then (Γ, Λ) is a HUP if and only if $J_k(r) \neq 0$ for all $k \in \mathbb{Z}_+$.

(ii) Let Λ be a straight line. Then (Γ, Λ) is not a HUP.

(iii) Let $\Lambda = L_1 \cup L_2$, where L_j; $j = 1, 2$ are two straight lines. If L_1 and L_2 are parallel, then (Γ, Λ) is a HUP.

(iv) Let L_j; $j = 1, 2, \ldots, N$ be the N different straight lines which intersect at one point and angle between any of two lines out of these N lines is of the form $\pi\alpha$. Let $\Lambda_N = \bigcup\limits_{j=1}^{N} L_j$. Then (Γ, Λ_N) is not a HUP if and only if α is rational.

In contrast to the case of finitely many straight lines, P. Sjolin [16] has shown that if $\Lambda = \bigcup\limits_{k=1}^{\infty} L_k$, where $\{L_k\}$ is a sequence of straight lines which intersect at one point. Then (S^1, Λ) is a HUP.

Remark 1.7. Since we know that any homogeneous harmonic polynomial on $\mathbb{R}^2$ can be expressed as $Ar^j \sin(j\theta + \delta)$ for some $j \in \mathbb{N}$ and $\delta \in [0, 2\pi)$ (see [5]), up to some rotation and translation, we can think of $\Lambda_N = \bigcup\limits_{k=1}^{N} L_k$, appeared in Theorem 1.6 (iv), as the zero set of some homogeneous harmonic polynomial. If (S^1, Λ) is a Heisenberg uniqueness pair, then the set Λ must be away from the zero set of any homogeneous harmonic polynomial. However, the converse is not true. Since (S^1, Λ) is not a HUP if Λ is a circle whose radius lie in the zero set of some Bessel function. Thus, it is an interesting question to examine the exceptional sets for the Heisenberg uniqueness pairs corresponding to circle.

Subsequently, some of the Heisenberg uniqueness pairs corresponding to the parabola have been obtained by P. Sjolin [17]. Let $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}$.

Theorem 1.8. [17] *Let Γ denote the parabola $y = x^2$.*

(i) Let $\Lambda = L$ be a straight line. Then (Γ, Λ) is a HUP if and only if L is parallel to the X-axis.

(ii) Let $\Lambda = L_1 \cup L_2$, where L_j; $j = 1, 2$ are two different straight lines. Then (Γ, Λ) is a HUP.

(iii) Let L_j; $j = 1, 2$ be two different straight lines which are not parallel to the X-axis. Let $E_j \subset L_j$ and $|E_j| > 0$; $j = 1, 2$. If $\Lambda = E_1 \cup E_2$, then (Γ, Λ) is a HUP.

The question of Heisenberg uniqueness pair in the higher dimension has been first taken up by F. J. Gonzalez Vieli [19, 20].

Theorem 1.9. [19] *Let $\Gamma = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and Λ a sphere of radius r. Then (Γ, Λ) is a HUP if and only if $J_{(n+2k-2)/2}(r) \neq 0$ for all $k \in \mathbb{Z}_+$.*

Theorem 1.10. [20] *Let Γ be the paraboloid $x_n = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$ in $\mathbb{R}^n$ and Λ an affine hyperplane in $\mathbb{R}^n$ of dimension $n-1$. Then (Γ, Λ) is a HUP if and only if Λ is parallel to the hyperplane $x_n = 0$.*

Let Γ denote a system of three parallel lines in the plane that can be expressed as $\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}$, where $\alpha < \beta < \gamma$ and $(\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}$. By the invariance properties of HUP, one can assume that $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$ with $p \geq 2$. The following characterization for the Heisenberg uniqueness pairs corresponding to the above mentioned three parallel lines has been given by D. B. Babot [2].

Theorem 1.11. [2] *Let $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$ with $p \geq 2$ and $\Lambda \subset \mathbb{R}^2$ a closed set which is 2-periodic with respect to the second variable. Then (Γ, Λ) is a HUP if and only if the set*

$$(1.2) \qquad \widetilde{\Lambda} = \Pi^3(\Lambda) \cup \left[\Pi^2(\Lambda) \smallsetminus \Pi^{2^*}(\Lambda)\right] \cup \left[\Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)\right]$$

is dense in $\mathbb{R}$.

For the notations appeared in Equation (1.2), we would like to refer the article [2], as those notations are quite involved. However, the nature of their occurrence can be understood in the beginning of Section 3 when we formulate the four lines problem.

Further, Jaming and Kellay [8] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to certain algebraic curves.

Theorem 1.12. [8] *Let Γ be any of the following curves:*
  (i) the graph of $\psi(t) = |t|^\alpha$, $t \in \mathbb{R}$, $\alpha > 0$;
 (ii) a hyperbola;
(iii) a polygon;
 (iv) an ellipse.
Then there exists a set $E \subset (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ of positive Lebesgue measure such that for $(\theta_1, \theta_2) \in E$, the pair $(\Gamma, L_{\theta_1} \cup L_{\theta_2})$ is a HUP.

2. A REVIEW OF THE HEISENBERG UNIQUENESS PAIRS FOR THE SPIRAL, HYPERBOLA, CIRCLE AND EXPONENTIAL CURVES

In this section, we will work out some of the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and certain exponential curves by using the basic tools of the Fourier analysis. Though, a complete characterization for the Heisenberg uniqueness pairs corresponding to either of the above curves is still open.

First, we prove that the spiral is a Heisenberg uniqueness pair for the anti-spiral.

Theorem 2.1. *Suppose* $\Gamma = \{(e^{-t}\cos t, e^{-t}\sin t) : t \geq 0\}$ *is a spiral and let* $\Lambda = \{(e^s\cos s, e^s\sin s) : s \leq 0\}$. *Then* (Γ, Λ) *is a Heisenberg uniqueness pair.*

In order to prove Theorem 2.1, we need the following results from [3, 4].

Theorem 2.2. [4] *Let h be a bounded measurable function and $g \in L^1(\mathbb{R}^n)$. If $h * g$ vanishes identically, then $\hat{h}$ vanishes on the support of $\hat{g}$.*

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0; \ j = 1, \ldots, n\}$. The following result had appeared in the article [3] by Bagchi and Sitaram, p. 421, as a part of the proof of Proposition 2.1.

Proposition 2.3. [3] *Let h be a non-zero bounded Borel measurable function which is supported on $\mathbb{R}^n_+$. Then supp $\hat{h} = \mathbb{R}^n$.*

Proof of Theorem 2.1. Since μ is absolutely continuous with respect to the arc length measure on Γ, by Radon-Nikodym theorem there exists $f \in L^1[0, \infty)$ such that $d\mu = \sqrt{2}f(t)e^{-t}dt$. Let $g(t) = \sqrt{2}f(t)e^{-t}$. Then by the finiteness of μ, it follows that $g \in L^1[0, \infty)$. By hypothesis, $\hat{\mu}|_\Lambda = 0$ implies

$$(2.1) \quad \hat{\mu}(\xi, \eta) = \int_0^\infty e^{-i\pi e^{-t}(\xi\cos t + \eta\sin t)}d\mu(t) = \int_0^\infty e^{-i\pi e^{(s-t)}\cos(t-s)}g(t)dt = 0$$

for all $(\xi, \eta) \in \Lambda$. Let $H(t) = e^{-i\pi e^t\cos t}\chi_{[0,\infty)}(t)$ and $G(t) = g(t)\chi_{(0,\infty)}(t)$. Then from (2.1), we get $\hat{\mu}(\xi, \eta) = (H * G)(s) = 0 \ \forall \ s \in \mathbb{R}$. In view of Theorem 2.2, we infer that supp $\hat{H} \subset Z(\hat{G})$, where $Z(\hat{G})$ denotes the zero set of $\hat{G}$. As H is a non-zero bounded Borel measurable function supported in $[0, \infty)$, by Proposition 2.3 it follows that supp $\hat{H} = \mathbb{R}$ and hence $\hat{G} = 0$. Thus, $\mu = 0$.

Next, we work out some of the Heisenberg uniqueness pairs corresponding to certain exponential curves in the plane. Though, the result is true for a large class of exponential curves, for the sake of simplicity we prove only for a particular one.

Theorem 2.4. *Let $\alpha : \mathbb{R} \to \mathbb{R}_+$ be the function defined by $\alpha(t) = e^{t^2}$ and let $\Gamma = \{(t, \alpha(t)) : \ t \in \mathbb{R}\}$.*
(i) If Λ is a straight line parallel to the X-axis. Then (Γ, Λ) is a HUP.
(ii) Let $\Lambda = L_1 \cup L_2$, where L_j; $j = 1, 2$ are any two straight lines parallel to the Y-axis. Then (Γ, Λ) is a HUP.

In order to prove Theorem 2.4, we need the following two important results about the uniqueness of Fourier transform. First, we state a result which can be found in Havin and Joricke [6], p. 36.

Lemma 2.5. [6] *If $\varphi \in L^1(\mathbb{R})$ is supported in $[0, \infty)$ and $\int_\mathbb{R} \log |\hat{\varphi}|\frac{dx}{1+x^2} = -\infty$, then $\varphi = 0$.*

As a consequence of Lemma 2.5 we prove the following result.

Lemma 2.6. *Let $g \in L^1(\mathbb{R})$ and $\alpha : \mathbb{R} \to \mathbb{R}_+$ be defined by $\alpha(t) = e^{t^2}$. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$(2.2) \qquad \int_{\mathbb{R}} e^{-i\pi y \alpha(t)} g(t) dt = 0$$

for all $y \in E$ if and only if g is an odd function.

Proof. The left hand side of Equation (2.2) can be expressed as

$$
\begin{aligned}
I &= \int_{-\infty}^{0} e^{-i\pi y \alpha(t)} g(t) dt + \int_{0}^{\infty} e^{-i\pi y \alpha(t)} g(t) dt \\
&= \int_{0}^{\infty} e^{-i\pi y \alpha(t)} (g(t) + g(-t)) dt \\
&= \int_{0}^{\infty} e^{-i\pi y \alpha(t)} F(t) dt,
\end{aligned}
$$

where $F(t) = g(t) + g(-t)$ for all $t \geq 0$. Clearly $F \in L^1(0, \infty)$ and hence by the change of variables $u = \alpha(t)$, we have

$$(2.3) \qquad I = \int_{1}^{\infty} e^{-i\pi x u} F(\sqrt{\log u}) \frac{du}{2u\sqrt{\log u}}.$$

Let $\varphi(u) = F(\sqrt{\log u})/2u\sqrt{\log u} \, \chi_{(1,\infty)}(u)$. Then $\varphi \in L^1(\mathbb{R})$ and from (2.3) we have $I = \hat{\varphi}(y) = 0$ for all $y \in E$. Since $\hat{\varphi}$ vanishes on the set E of positive Lebesgue measure, by Lemma 2.5 it follows that $\varphi = 0$. That is, $F = 0$ and hence g is an odd function.

Conversely, if g is an odd function, then (2.2) trivially holds. $\qquad \square$

Proof of Theorem 2.4. (i) Since μ is supported on $\Gamma = \{(t, e^{t^2}) : t \in \mathbb{R}\}$, there exists $f \in L^1(\mathbb{R})$ such that $d\mu = f(t)\sqrt{1 + 4t^2 e^{2t^2}} \, dt$. Let $g(t) = f(t)\sqrt{1 + 4t^2 e^{2t^2}}$. Then by the finiteness of μ, it follows that $g \in L^1(\mathbb{R})$ and

$$\hat{\mu}(x, y) = \int_{\mathbb{R}} e^{-i\pi\left(xt + ye^{t^2}\right)} g(t) dt.$$

In view of the invariance property (i), we can assume that Λ is the X-axis. Hence $\hat{\mu}|_\Lambda = 0$ implies that $\hat{g}(x) = 0$ for all $x \in \mathbb{R}$. Thus, we conclude that $\mu = 0$.

(ii) By invariance property (i), we can assume L_1 is the Y-axis and L_2 the line $x = x_o$, where $x_o \neq 0$. Since $\hat{\mu}$ vanishes on L_1, by Lemma 2.6 it follows that g is odd. Also, $\hat{\mu}$ vanishes on the line L_2, implies that

$$\int_{\mathbb{R}} e^{-i\pi(x_o t + ye^{t^2})} g(t) dt = 0$$

for all $y \in \mathbb{R}$. In view of Lemma 2.6, it follows that $e^{-i\pi x_o t} g(t)$ is an odd function. Hence $e^{-i\pi x_o t} g(t) = -e^{i\pi x_o t} g(-t)$. Since g is odd, it implies that

$(e^{2i\pi x_o t} - 1)g(t) = 0$. As the identity $e^{2i\pi x_o t} = 1$ holds only for the countably many t, we conclude that $g = 0$. Thus, $\mu = 0$.

Remark 2.7. Let $\alpha : \mathbb{R} \to \mathbb{R}_+$ be an even smooth function having finitely many local extrema and $\Gamma = \{(t, \alpha(t)) : t \in \mathbb{R}\}$. Then the conclusions of Theorem 2.4 would also hold.

Next, we work out some of the Heisenberg uniqueness pairs corresponding to the circle. We show that (circle, spiral) is a HUP.

Let $\Gamma = S^1$ denote the unit circle in $\mathbb{R}^2$. If for $f \in L^1(\Gamma)$, we write $f(\theta)$ instead of $f(e^{i\theta})$, then f is a 2π periodic function and $f \in L^1[0, 2\pi]$. Let μ be a finite complex-valued Borel measure in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. Then there exists $f \in L^1(S^1)$ such that $d\mu = f(\theta)d\theta$. Now, we prove the following result.

Theorem 2.8. *Let $\Gamma = S^1$ and $\Lambda = \{(e^t \cos t, e^t \sin t) : t \leq 0\}$ be the spiral. Then (Γ, Λ) is a Heisenberg uniqueness pair.*

Proof. Since μ is supported on the unit circle Γ, we can write the Fourier transform of μ by

$$\hat{\mu}(x, y) = \int_{-\pi}^{\pi} e^{-i\pi(x\cos\theta + y\sin\theta)} f(\theta)d\theta.$$

Hence $\hat{\mu}$ can be extended holomorphically to $\mathbb{C}^2$. Thus, the function F defined by

$$F(z_1, z_2) = \int_{-\pi}^{\pi} e^{-i\pi(z_1\cos\theta + z_2\sin\theta)} f(\theta)d\theta,$$

is holomorphic on $\mathbb{C}^2$. In particular, $\hat{\mu} = F|_{\mathbb{R}^2}$ is a real analytic function. Since $\hat{\mu}$ vanishes on the spiral Λ, for any line L which passes through the origin, $\hat{\mu}|_{\Lambda \cap L} = 0$. As $(0, 0)$ is a limit point of the set $\Lambda \cap L$, it follows that $\hat{\mu}|_L = 0$. Since L is arbitrary, we infer that $\hat{\mu}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

Let $S_r = \{(r\cos t, r\sin t) : 0 \leq t < 2\pi\}$, where $J_k(r) \neq 0$ for all $k \in \mathbb{Z}$. Then $\hat{\mu}(r\cos t, r\sin t) = 0$ implies $h * f(t) = 0$, where $h(t) = e^{-i\pi r\cos t}$. As we know that the Fourier coefficients of h satisfying $\hat{h}(k) = i^k(-1)^k J_k(r)$, it follows that $\hat{f}(k)J_k(r) = 0$ for all $k \in \mathbb{Z}$. Since $J_k(r) \neq 0$ for all $k \in \mathbb{Z}$, $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ and hence $f = 0$. $\square$

Remark 2.9. A set which is determining set for any real analytic function is called NA - set. For instance, the spiral is an NA - set in the plane (see [13]). If μ is a finite Borel measure supported on a closed and bounded curve Γ, then $\hat{\mu}$ is real analytic. Thus, $(\Gamma, NA$ - set$)$ is a Heisenberg uniqueness pair. However, the converse is not true.

Hence, in view of Remarks 1.7 and 2.9 we expect that the exceptional sets for the Heisenberg uniqueness pairs corresponding to the unit circle $\Gamma = S^1$ are eventually contained in the zero sets of all homogeneous harmonic polynomials union with the countably many circles whose radii are lying in the zero set of

the certain class of Bessel functions. On the basis of these credible observations, we are trying to find out a complete characterization of the Heisenberg uniqueness pairs corresponding to circle which may be presented somewhere else.

Next, we work out some of the Heisenberg uniqueness pairs corresponding to the hyperbola. Though in this case, Hedenmalm and Montes-Rodrguez [7] have found that some discrete set $\Lambda_{\alpha,\beta}$ is enough for $(\Gamma, \Lambda_{\alpha,\beta})$ to be a Heisenberg uniqueness pair. However, our approach is to consider those sets Λ which are essentially a union of continuous curves and located somewhere else than the set $\Lambda_{\alpha,\beta}$.

Theorem 2.10. *Let* $\Gamma = \{(\cosh t, \sinh t) : t \geq 0\}$ *be a branch of the hyperbola and* $\Lambda = \{(\cosh s, -\sinh s) : s \in \mathbb{R}\}$. *Then* (Γ, Λ) *is a HUP.*

Proof. Since μ is supported on Γ, there exists $f \in L^1[0,\infty)$ such that $d\mu = f(t)\sqrt{\cosh 2t}\, dt$. If we write $g(t) = f(t)\sqrt{\cosh 2t}$, then $g \in L^1[0,\infty)$ and

$$\hat{\mu}(x, y) = \int_0^\infty e^{-i\pi(x\cosh t + y\sinh t)} g(t)dt.$$

By hypothesis, $\hat{\mu}|_\Lambda = 0$ implies

$$(2.4) \qquad \hat{\mu}(x, y) = \int_0^\infty e^{-i\pi \cosh(t-s)} g(t)dt = 0$$

for all $(x, y) \in \Lambda$. Let $H(t) = e^{-i\pi \cosh t}\chi_{[0,\infty)}(t)$ and $G(t) = g(t)\chi_{(0,\infty)}(t)$. Then from (2.4) we get $\hat{\mu}(x, y) = (H * G)(s) = 0$ for all $s \in \mathbb{R}$. In view of Theorem 2.2, it follows that supp $\hat{H} \subset Z(\hat{G})$. Hence by Proposition 2.3, supp $\hat{H} = \mathbb{R}$. Thus, we conclude that $G = 0$. $\qquad\square$

Theorem 2.11. *Let* $\Gamma = \{(\cosh t, \sinh t) : t \in \mathbb{R}\}$ *and* $\Lambda = L_1 \cup L_2$, *where* L_j; $j = 1, 2$ *are any two lines parallel to the X-axis. Then* (Γ, Λ) *is a HUP.*

We need the following result in order to prove Theorem 2.11.

Lemma 2.12. *Let* $g \in L^1(\mathbb{R})$ *and* $E \subset \mathbb{R}$ *such that* $|E| > 0$. *Then*

$$(2.5) \qquad \int_{\mathbb{R}} e^{-i\pi x \cosh t} g(t)dt = 0$$

for all $x \in E$ *if and only if* g *is an odd function.*

Proof. The left-hand side of Equation (2.5) can be expressed as

$$
\begin{aligned}
I &= \int_{-\infty}^0 e^{-i\pi x \cosh t} g(t)dt + \int_0^\infty e^{-i\pi x \cosh t} g(t)dt \\
&= \int_0^\infty e^{-i\pi x \cosh t}(g(t) + g(-t))dt \\
&= \int_0^\infty e^{-i\pi x \cosh t} F(t)dt,
\end{aligned}
$$

where $F(t) = g(t) + g(-t)$ for all $t \geq 0$. Clearly $F \in L^1(0, \infty)$. By change of variables $u = \cosh t$, we get

$$(2.6) \qquad I = \int_1^\infty e^{-i\pi x u} F(\cosh^{-1} u) \frac{du}{\sqrt{u^2 - 1}}.$$

If we substitute $\varphi(u) = F(\cosh^{-1} u)/\sqrt{u^2 - 1} \, \chi_{(1,\infty)}$, then $\varphi \in L^1(\mathbb{R})$ and $I = \hat{\varphi}(x) = 0$ for all $x \in E$. Hence by Lemma 2.5, it follows that $\varphi = 0$. Thus, we infer that g is an odd function.

Conversely, suppose g is an odd function, then (2.5) trivially holds. $\square$

Proof of Theorem 2.11. By invariance property (i), we can assume that L_1 is the X-axis and L_2 the line $y = y_o$, where $y_o \neq 0$. Since μ is supported on the hyperbola Γ, there exists $f \in L^1(\mathbb{R})$ such that $d\mu = f(t)\sqrt{\cosh 2t} \, dt$. Let $g(t) = f(t)\sqrt{\cosh 2t}$, then $g \in L^1(\mathbb{R})$. Hence in view of Lemma 2.12, $\hat{\mu}$ vanishes on L_1 implies that g is an odd function. Further, $\hat{\mu}|_{L_2} = 0$ implies that

$$\int_{\mathbb{R}} e^{-i\pi(x \cosh t + y_o \sinh t)} g(t) dt = 0$$

for all $x \in \mathbb{R}$. Then by Lemma 2.12 the function $e^{-i\pi y_o \sinh t} g(t)$ will be an odd function. Hence $e^{-i\pi y_o \sinh t} g(t) = -e^{i\pi y_o \sinh t} g(-t)$. As g is an odd function, it follows that $\left(e^{2i\pi y_o \sinh t} - 1 \right) g(t) = 0$. Using the fact that $e^{2i\pi y_o \sinh t} = 1$ holds only for the countably many values of t, we conclude that $g = 0$.

Theorem 2.13. *Let $\Gamma = \{(\cosh t, \sinh t) : t \in \mathbb{R}\}$ and $\Lambda = L_1 \cup L_2$, where L_j; $j = 1, 2$ are any two straight lines which intersect at an angle $\alpha \in (0, \frac{\pi}{4})$. Then (Γ, Λ) is a HUP.*

Proof. Without loss of generality, we can assume that L_1 is the X-axis and $L_2 = \{(s \cosh t_o, -s \sinh t_o) : s \in \mathbb{R}\}$, where $\tan \alpha = -\tanh t_o$. Since μ is supported on the hyperbola Γ, as similar to Theorem 2.11 there exists $g \in L^1(\mathbb{R})$ such that $d\mu = g(t) dt$. Suppose $\hat{\mu} = 0$ on Λ, then we have

$$\int_{\mathbb{R}} e^{-i\pi(x \cosh t + y \sinh t)} g(t) dt = 0$$

for all $(x, y) \in L_2$. This in turn implies

$$\int_{\mathbb{R}} e^{-i\pi s \cosh t} g(t + t_o) dt = 0$$

for all $s \in \mathbb{R}$. In view of Lemma 2.12, it follows that $g(t_o + \cdot)$ must be an odd function. Since $\hat{\mu}$ is also vanishing on the X-axis, g will be odd. Hence $g(2t_o \pm t) = g(t)$ for all $t \in \mathbb{R}$. That is, g is a periodic function contained in $L^1(\mathbb{R})$. Thus, we conclude that $g = 0$. $\square$

Remark 2.14. (a). Let Γ be the hyperbola and Λ a straight line parallel to the X-axis. Then (Γ, Λ) is not a HUP. Consider $g = \sqrt{\cosh 2t} \sin t \, \chi_{(-\pi,\pi)}$ and $d\mu = g(t) dt$. Then $\hat{\mu}$ vanishes on Λ.

(**b**). We would like to mention that Theorem 2.13 is contained well in the case (ii) of Theorem 1.12 due to Jaming and Kellay [8]. However, our approach for proof of Theorem 2.13 is quite different.

3. Heisenberg uniqueness pairs corresponding to the four parallel lines

A characterization of the Heisenberg uniqueness pairs corresponding to any two parallel straight lines have been done by Hedenmalm et al. [7]. Further, D. B. Babot [2] has worked out an analogous result for a certain system of three parallel lines. In this section, we prove a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of four parallel lines. In the above case, we observe the phenomenon of interlacing of three totally disconnected sets.

Let Γ_o denote a system of four parallel lines that can be expressed as $\Gamma_o = \mathbb{R} \times \{\alpha, \beta, \gamma, \delta\}$, where $\alpha < \beta < \gamma < \delta$, $p = (\delta - \alpha)/(\beta - \alpha) \in \mathbb{N} \smallsetminus \{1, 2\}$ and $(\gamma - \alpha)/(\beta - \alpha) = 2$. If (Γ_o, Λ_o) is a HUP, then by using invariance property (i), (Γ_o, Λ_o) can be reduced to $(\Gamma_o - (0, \alpha), \Lambda_o)$. Since scaling can be thought as a diagonal matrix, by using invariance property (ii), $(\Gamma_o - (0, \alpha), \Lambda_o)$ can be reduced to $(T^{-1}(\Gamma_o - (0, \alpha)), T^*\Lambda_o)$, where $T = \mathrm{diag}\{(\beta - \alpha), (\beta - \alpha)\}$. Let $\Lambda = T^*\Lambda_o$ and $\Gamma = T^{-1}(\Gamma_o - (0, \alpha))$. Then $\Gamma = \mathbb{R} \times \{0, 1, 2, p\}$, where $p \in \mathbb{N}$ with $p \geq 3$. Thus, (Γ_o, Λ_o) is a HUP if and only if (Γ, Λ) is a HUP.

Before we state our main result of this section, we need to set up some necessary notations and the subsequent auxiliary results.

Let μ be a finite Borel measure which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. Then there exist functions $f_k \in L^1(\mathbb{R})$; $k = 0, 1, 2, 3$ such that

(3.1) $d\mu = f_0(x)dxd\delta_0(y) + f_1(x)dxd\delta_1(y) + f_2(x)dxd\delta_2(y) + f_3(x)dxd\delta_p(y),$

where δ_t denotes the point mass measure at t. By taking the Fourier transform of both sides of (3.1) we get

(3.2) $\hat{\mu}(\xi, \eta) = \hat{f}_0(\xi) + e^{\pi i \eta}\hat{f}_1(\xi) + e^{2\pi i \eta}\hat{f}_2(\xi) + e^{p\pi i \eta}\hat{f}_3(\xi).$

Notice that for each fixed $(\xi, \eta) \in \Lambda$, the right-hand side of Equation (3.2) is a trigonometric polynomial of degree p that could have preferably some missing terms. Therefore, it is an interesting question to find out the smallest set Λ that determines the above trigonometric polynomial. We observe that the size of Λ depends on the choice of a number of lines as well as irregular separation among themselves. That is, a larger number of lines or value of p would force smaller size of Λ. Eventually, the problem would become immensely difficult for a large value of p.

Observe that $\hat{\mu}$ is a 2-periodic function in the second variable. Hence, for any set $\Lambda \subset \mathbb{R}^2$, it is enough to consider the set

$$\mathcal{L}(\Lambda) = \{(\xi, \eta) : (\xi, \eta + 2k) \in \Lambda, \text{ for some } k \in \mathbb{Z}\}$$

for the purpose of HUP. Also, it is easy to verify that (Γ, Λ) is a HUP if and only if $(\Gamma, \overline{\mathcal{L}(\Lambda)})$ is a HUP, where $\overline{\mathcal{L}(\Lambda)}$ denotes the closure of $\mathcal{L}(\Lambda)$ in $\mathbb{R}^2$. In view of the above facts, it is enough to work with the closed set $\Lambda \subset \mathbb{R}^2$ which is 2-periodic with respect to the second variable.

Now, it is evident from the Riemann-Lebesgue lemma that the exponential functions, which appeared in (3.2), cannot be expressed as the Fourier transform of functions in $L^1(\mathbb{R})$. However, they can locally agree with the Fourier transform of functions in $L^1(\mathbb{R})$. Hence, in view of the condition $\hat{\mu}|_{\Lambda} = 0$, we can classify these related exponential functions.

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let I_ξ denote an interval containing ξ. We define three functions spaces in the following way.

(A). $L_{loc}^{E,\xi} = \{\psi : E \to \mathbb{C}$ such that $\psi(\xi) \neq 0$ and there is an interval I_ξ and a function $\varphi \in L^1(\mathbb{R})$ which satisfies $\psi = \hat{\varphi}$ on $I_\xi \cap E\}$.

(B). $P^{1,2}[L_{loc}^{E,\xi}] = \{\psi : E \to \mathbb{C}$ such that there is an interval I_ξ and $\varphi_j \in L^1(\mathbb{R})$; $j = 0, 1$ which satisfies $\psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E\}$.

Now, for $p \in \mathbb{N}$ with $p \geq 3$, we define the third functions space as follows.

(C). $P^{1,p}[L_{loc}^{E,\xi}] = \{\psi : E \to \mathbb{C}$ such that there is an interval I_ξ and functions $\varphi_j \in L^1(\mathbb{R})$; $j = 0, 1, 2$ which satisfy $\psi^p + \hat{\varphi}_2 \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E\}$.

We will frequently use the following Wiener's lemma that plays a key role in the rest part of the arguments for proofs.

Lemma 3.1. [9] *Let $\psi \in L_{loc}^{E,\xi}$ and $\psi(\xi) \neq 0$. Then $1/\psi \in L_{loc}^{E,\xi}$.*

For more details, see [9], p.57.

In view of Lemma 3.1, we derive the following relation among the sets which are described by **(A)**, **(B)** and **(C)**. We would like to mention that the integral choice of p in Lemma 3.2 has been considered for a convenience.

Lemma 3.2. *For $p \geq 3$, the following inclusions hold.*

$$(3.3) \qquad L_{loc}^{E,\xi} \subset P^{1,2}[L_{loc}^{E,\xi}] \subset P^{1,p}[L_{loc}^{E,\xi}].$$

Proof. (**a**) If $\psi \in L_{loc}^{E,\xi}$, then by the Wiener's lemma $1/\psi \in L_{loc}^{E,\xi}$. By definition, there exist intervals I_1, I_2 containing ξ and functions $f, g \in L^1(\mathbb{R})$ such that $\psi = \hat{f}$ on $I_1 \cap E$ and $\frac{1}{\psi} = \hat{g}$ on $I_2 \cap E$. Hence we can extract an interval $I_3 \subset I_1 \cap I_2$ containing ξ such that $\psi^2 = \frac{\hat{f}}{\hat{g}}$ on $I_3 \cap E$. As $\hat{g}(\xi) \neq 0$, there exists an interval I_4 containing ξ and a function $h \in L^1(\mathbb{R})$ such that $\frac{1}{\hat{g}} = \hat{h}$ on $I_4 \cap E$. Further, we can extract an interval $I_5 \subset I_3 \cap I_4$ containing ξ such that

$$(3.4) \qquad \psi^2 = \hat{f}\, \hat{h} = \widehat{f * h} = \hat{\varphi}$$

on $I_5 \cap E$, where $\varphi = f * h \in L^1(\mathbb{R})$. This implies $\psi^2 \in L_{loc}^{E,\xi}$. Hence by the induction argument, it can be shown that $\psi^p \in L_{loc}^{E,\xi}$, whenever $p \in \mathbb{N}$. Now,

consider a function $f_0 \in L^1(\mathbb{R})$ such that $I_1 \subset$ supp $\hat{f}_0$. Since $\psi = \hat{f}$ on $I_1 \cap E$, it follows that

$$(3.5) \qquad\qquad \hat{f}_0 \psi = \hat{f}_0 \hat{f} = \widehat{f_0 * f}$$

on $I_1 \cap E$. Hence from (3.4) and (3.5) we conclude that

$$(3.6) \qquad\qquad \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$$

on $I_\xi \cap E$, where $I_\xi \subset I_1 \cap I_5$, $\varphi_0 = -(f_0 * f + \varphi)$ and $\varphi_1 = f_0$. Thus, $\psi \in P^{1,2}[L_{loc}^{E,\xi}]$. By applying induction, we can show that $\psi^p + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$, whenever $p \in \mathbb{N}$.

(**b**) If $\psi \in P^{1,2}[L_{loc}^{E,\xi}]$, then there exists an interval I_ξ containing ξ and functions $f, g \in L^1(\mathbb{R})$ such that

$$(3.7) \qquad\qquad \psi^2 + \hat{f}\psi + \hat{g} = 0$$

on $I_\xi \cap E$. Now, consider a function $f_0 \in L^1(\mathbb{R})$ such that $I_\xi \subset$ supp $\hat{f}_0$. After multiplying (3.7) by ψ and $\hat{f}_0$ separately and adding the resultant equations, we can write

$$\psi^3 + \left(\hat{f}_0 + \hat{f}\right)\psi^2 + \left(\hat{f}_0\hat{f} + \hat{g}\right)\psi + \hat{f}_0\hat{g} = 0.$$

Hence for the appropriate choice of φ_j; $j = 0, 1, 2$, we have

$$(3.8) \qquad\qquad \psi^3 + \hat{\varphi}_2 \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$$

on $I_\xi \cap E$. Further by induction, it follows that $\psi^p + \hat{\varphi}_2 \psi^2 + \hat{\varphi}_1 \psi + \hat{\varphi}_0 = 0$ on $I_\xi \cap E$, whenever $p \in \mathbb{N}$. Thus, $\psi \in P^{1,p}[L_{loc}^{E,\xi}]$. $\qquad\square$

Let $\Pi(\Lambda)$ be the projection of Λ on $\mathbb{R} \times \{0\}$. For $\xi \in \Pi(\Lambda)$, we denote the corresponding image on the η - axis by

$$\Sigma_\xi = \{\eta \in [0, 2) : (\xi, \eta) \in \Lambda\}.$$

Now, we require analyzing the set $\Pi(\Lambda)$ to know its basic geometrical structure in accordance with the Heisenberg uniqueness pair. Since it is expected that the set Σ_ξ may consist one or more image points depending upon the order of its winding, the set $\Pi(\Lambda)$ can be decomposed into the following four disjoint sets. For the sake of convenience, we denote $F_o = \{0, 1, 2, 3\}$.

(**P$_1$**). $\Pi^1(\Lambda) = \{\xi \in \Pi(\Lambda) : \text{ there is a unique } \eta_0 \in \Sigma_\xi\}$.

(**P$_2$**). $\Pi^2(\Lambda) = \{\xi \in \Pi(\Lambda) : \text{ there are only two distinct } \eta_j \in \Sigma_\xi; \ j = 0, 1\}$.

In order to describe the rest of the two partitioning sets, we will use the notion of symmetric polynomial. For each $k \in \mathbb{Z}_+$, the complete homogeneous symmetric polynomial H_k of degree k is the sum of all monomials of degree k. That is,

$$H_k(x_1, \ldots, x_n) = \sum_{l_1 + \cdots + l_n = k; \ l_i \geq 0} x_1^{l_1} \ldots x_n^{l_n}.$$

For more details, we refer to [11].

Consider four distinct image points $\eta_j \in [0, 2)$ and denote $a_j = e^{\pi i \eta_j}$; $j \in F_0$. For $p \geq 3$, we define the remaining two sets as follows:

($\mathbf{P_3}$). $\Pi^3(\Lambda) = \{\xi \in \Pi(\Lambda) : \text{there are at least three distinct } \eta_j \in \Sigma_\xi \text{ for } j = 0, 1, 2 \text{ and if there is another } \eta_3 \in \Sigma_\xi, \text{ then } H_{p-2}(a_0, a_1, a_2) = H_{p-2}(a_0, a_1, a_3)\}$.

($\mathbf{P_4}$). $\Pi^4(\Lambda) = \{\xi \in \Pi(\Lambda) : \text{there are at least four distinct } \eta_j \in \Sigma_\xi; \ j \in F_o \text{ which satisfy } H_{p-2}(a_0, a_1, a_2) \neq H_{p-2}(a_0, a_1, a_3)\}$.

In this way, we get the desired decomposition as $\Pi(\Lambda) = \bigcup\limits_{j=1}^{4} \Pi^j(\Lambda)$.

Now, for three distinct image points $\eta_j \in [0, 2)$; $j = 0, 1, 2$, denote $a = e^{\pi i \eta_0}$, $b = e^{\pi i \eta_1}$ and $c = e^{\pi i \eta_2}$. Consider the system of equations $A_\xi^3 X = B_\xi^p$, where

$$(3.9) \qquad A_\xi^3 = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix},$$

$X_\xi = (\tau_0, \tau_1, \tau_2)$ and $B_\xi^p = -(a^p, b^p, c^p)$. Since $\det A_\xi^3 = (a-b)(b-c)(c-a) \neq 0$, $A_\xi^3 X = B_\xi^p$ has a unique solution. A simple calculation gives

$$
\begin{aligned}
(3.10) \quad \tau_o &= -abc H_{p-3}(a, b, c), \\
\tau_1 &= H_{p-1}(a, b, c) - \left(a^{p-1} + b^{p-1} + c^{p-1}\right) + \sum_{\substack{l+m+n=p-1 \\ l,m,n \geq 1}} a^l b^m c^n, \\
\tau_2 &= -H_{p-2}(a, b, c).
\end{aligned}
$$

Since measure in the question is supported on a certain system of four parallel lines and the exponential functions which have appeared in (3.2) can locally agree with the Fourier transform of some functions in $L^1(\mathbb{R})$, the following sets sitting in $\Pi(\Lambda)$ seems to be dispensable in the process of getting the Heisenberg uniqueness pairs.

($\mathbf{P_{1^*}}$). As each $\xi \in \Pi^1(\Lambda)$ has a unique image in Σ_ξ, we can define a function χ_0 on $\Pi^1(\Lambda)$ by $\chi_0(\xi) = e^{\pi i \eta_0}$, where $\eta_0 = \eta_0(\xi) \in \Sigma_\xi$. Now, the first dispensable set can be defined by

$$\Pi^{1^*}(\Lambda) = \left\{\xi \in \Pi^1(\Lambda) : \chi_0 \in P^{1,p}[L_{loc}^{\Pi^1(\Lambda),\xi}]\right\}.$$

Next, for $\xi \in \Pi^2(\Lambda)$, let $\chi_j(\xi) = e^{\pi i \eta_j}$, where $\eta_j = \eta_j(\xi) \in \Sigma_\xi$; $j = 0, 1$.

($\mathbf{P'_{2^*}}$). Since each $\xi \in \Pi^2(\Lambda)$ has two distinct image points in Σ_ξ, we define two functions δ_j on $\Pi^2(\Lambda)$; $j = 0, 1$ such that $X_\xi = (\delta_0(\xi), \delta_1(\xi))$ is the solution of $A_\xi^2 X_\xi = B_\xi^2$, where

$$A_\xi^2 = \begin{pmatrix} 1 & \chi_0(\xi) \\ 1 & \chi_1(\xi) \end{pmatrix}$$

and $B_\xi^2 = -\left(\chi_0(\xi)^2, \chi_1(\xi)^2\right)$. In this way, an auxiliary dispensable set can be defined by

$$\Pi^{2^*}(\Lambda) = \left\{ \xi \in \Pi^2(\Lambda) : \delta_j \in L_{loc}^{\Pi^2(\Lambda),\xi}; \ j = 0,1 \right\}.$$

($\mathbf{P_{2*}''}$). Further, we define three functions ρ_j on $\Pi^2(\Lambda)$; $j = 0,1,2$ such that $X_\xi = (\rho_0(\xi), \rho_1(\xi), \rho_2(\xi))$ becomes a solution of $A_\xi^p X_\xi = B_\xi^p$, where

$$A_\xi^p = \begin{pmatrix} 1 & \chi_0(\xi) & \chi_0(\xi)^2 \\ 1 & \chi_1(\xi) & \chi_1(\xi)^2 \end{pmatrix}$$

and $B_\xi^p = -(\chi_0(\xi)^p, \chi_1(\xi)^p)$. Hence the second dispensable set can be defined by

$$\Pi_{2*}^p(\Lambda) = \left\{ \xi \in \Pi^2(\Lambda) : \rho_j \in L_{loc}^{\Pi^2(\Lambda),\xi}; \ j = 0,1,2 \right\}.$$

For $\xi \in \Pi^3(\Lambda)$, let $\chi_j(\xi) = e^{\pi i \eta_j}$, where $\eta_j = \eta_j(\xi) \in \Sigma_\xi$; $j = 0,1,2$.

($\mathbf{P_{3*}'}$). For each $\xi \in \Pi^3(\Lambda)$ has three distinct image points in Σ_ξ, we define three functions e_j on $\Pi^3(\Lambda)$; $j = 0,1,2$ such that $X_\xi = (e_0(\xi), e_1(\xi), e_2(\xi))$ is the solution of $A_\xi^3 X_\xi = B_\xi^3$, where A_ξ^3 is the matrix given by Equation (3.9) and $B_\xi^3 = -\left(\chi_0(\xi)^3, \chi_1(\xi)^3, \chi_2(\xi)^3\right)$. Hence another auxiliary dispensable set can be defined by

$$\Pi^{3^*}(\Lambda) = \left\{ \xi \in \Pi^3(\Lambda) : e_j \in L_{loc}^{\Pi^3(\Lambda),\xi}; \ j = 0,1,2 \right\}.$$

($\mathbf{P_{3*}''}$). Once again we define three functions τ_j on $\Pi^3(\Lambda)$; $j = 0,1,2$ such that $X_\xi = (\tau_0(\xi), \tau_1(\xi), \tau_2(\xi))$ is the solution of $A_\xi^3 X_\xi = B_\xi^p$, where $B_\xi^p = -(\chi_0(\xi)^p, \chi_1(\xi)^p, \chi_2(\xi)^p)$. Hence the third dispensable set can be defined by

$$\Pi_{3*}^p(\Lambda) = \left\{ \xi \in \Pi^3(\Lambda) : \tau_j \in L_{loc}^{\Pi^3(\Lambda),\xi}; \ j = 0,1,2 \right\}.$$

Now, we prove the following two lemmas that speak about a sharp contrast in the pattern of dispensable sets as compared to dispensable sets which appeared in two lines and three lines results. That is, a larger value of p will increase the size of dispensable sets in case of four lines problem. Further, we observe that dispensable sets are eventually those sets contained in $\Pi(\Lambda)$ where we could not solve Equation (3.2). For more details, we refer to [2, 7].

Lemma 3.3. *For $p \geq 3$, the following inclusion holds.*

$$\Pi^{2^*}(\Lambda) \subset \Pi_{2*}^p(\Lambda).$$

Proof. If $\xi_o \in \Pi^{2^*}(\Lambda)$, then $\delta_j \in L_{loc}^{\Pi^2(\Lambda),\xi_o}$. Hence there exists an interval I_{ξ_o} containing ξ_o and $\varphi_j \in L^1(\mathbb{R})$ such that $\delta_j = \hat{\varphi}_j$; $j = 0,1$ satisfy

$$\hat{\varphi}_0 + \hat{\varphi}_1 \chi_j + {\chi_j}^2 = 0$$

on $I_{\xi_o} \cap \Pi^2(\Lambda)$, whenever $j = 0, 1$. Now, by the similar iteration as in the proof of Lemma 3.2(b), we infer that there exist a common set of $\psi_j \in L^1(\mathbb{R})$; $j = 0, 1, 2$ such that

$$\hat{\psi}_0 + \hat{\psi}_1 \chi_j + \hat{\psi}_2 {\chi_j}^2 + {\chi_j}^p = 0$$

on $I_{\xi_o} \cap \Pi^2(\Lambda)$, whenever $j = 0, 1$. If we denote $\hat{\psi}_j = \rho_j$, then it is easy to see that $\xi_o \in \Pi^p_{2*}(\Lambda)$. □

Lemma 3.4. *For $p \geq 3$, the following inclusion holds.*

$$\Pi^{3^*}(\Lambda) \subseteq \Pi^p_{3*}(\Lambda).$$

Moreover, equality holds for $p = 3$.

Proof. If $\xi_o \in \Pi^{3^*}(\Lambda)$, then $e_j \in L_{loc}^{\Pi^3(\Lambda), \xi_o}$. Hence there exists an interval I_{ξ_o} containing ξ_o and $\varphi_j \in L^1(\mathbb{R})$ such that $e_j = \hat{\varphi}_j$; $j = 0, 1, 2$ satisfy

$$\hat{\varphi}_0 + \hat{\varphi}_1 \chi_j + \hat{\varphi}_2 {\chi_j}^2 + {\chi_j}^3 = 0$$

on $I_{\xi_o} \cap \Pi^3(\Lambda)$, whenever $j = 0, 1, 2$. By the similar iteration as in the proof of Lemma 3.2(b), it follows that there exist $\psi_j \in L^1(\mathbb{R})$; $j = 0, 1, 2$ such that

$$\hat{\psi}_0 + \hat{\psi}_1 \chi_j + \hat{\psi}_2 {\chi_j}^2 + {\chi_j}^p = 0$$

on $I_{\xi_o} \cap \Pi^3(\Lambda)$, whenever $j = 0, 1, 2$. If we denote $\hat{\psi}_j = \tau_j$, then it is easy to verify that $\xi_o \in \Pi^p_{3*}(\Lambda)$. □

On the basis of structural properties of the dispensable sets, we observe that these sets are essentially minimizing the size of projection $\Pi(\Lambda)$. Now, we can state our main result of this section about the Heisenberg uniqueness pairs corresponding to the above described system of four parallel straight lines.

Theorem 3.5. *Let $\Gamma = \mathbb{R} \times \{0, 1, 2, p\}$, where $p \in \mathbb{N}$ and $p \geq 3$. Let $\Lambda \subset \mathbb{R}^2$ be a closed set which is 2-periodic with respect to the second variable. Suppose $\Pi(\Lambda)$ is dense in $\mathbb{R}$. If (Γ, Λ) is a Heisenberg uniqueness pair, then the set*

$$\widetilde{\Pi}(\Lambda) = \Pi^4(\Lambda) \bigcup_{j=0}^{2} \left[\Pi^{(3-j)}(\Lambda) \smallsetminus \Pi^{(3-j)^*}(\Lambda) \right]$$

is dense in $\mathbb{R}$. Conversely, if the set

$$\widetilde{\Pi_p}(\Lambda) = \Pi^4(\Lambda) \cup \left[\Pi^3(\Lambda) \smallsetminus \Pi^p_{3*}(\Lambda) \right] \cup \left[\Pi^2(\Lambda) \smallsetminus \Pi^p_{2*}(\Lambda) \right] \cup \left[\Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda) \right]$$

is dense in $\mathbb{R}$, then (Γ, Λ) is a Heisenberg uniqueness pair.

Remark 3.6. In view of Lemma 3.3, we infer that $\Pi^{2^*}(\Lambda)$ is a proper subset of $\Pi^p_{2*}(\Lambda)$ for any $p \geq 3$. However, for $p = 3$, Lemma 3.4 yields $\Pi^{3^*}(\Lambda) = \Pi^p_{3*}(\Lambda)$. Hence for any $p \geq 3$, the set $\widetilde{\Pi_p}(\Lambda)$ is properly contained in $\widetilde{\Pi}(\Lambda)$. Thus, an analogous result for four lines problem as compared to three lines result is still open.

We need the following two lemmas which are required to prove the necessary part of Theorem 3.5. The main idea behind these lemmas is to pull down an interval from some of the partitioning sets of the projection $\Pi(\Lambda)$. The above argument helps to negate the assumption that $\widetilde{\Pi}(\Lambda)$ is not dense in $\mathbb{R}$.

Lemma 3.7. *Suppose I is an interval such that $I \cap \Pi^{2^*}(\Lambda)$ is dense in I. Then there exists an interval $I' \subset I$ such that $I' \subset \bigcup\limits_{j=2}^{4} \Pi^j(\Lambda)$.*

Proof. If $\bar{\xi} \in I \cap \Pi^{2^*}(\Lambda)$, then $\delta_j \in L_{loc}^{\Pi^2(\Lambda),\bar{\xi}}$; $j = 0, 1$. By hypothesis, $I \cap \Pi^{2^*}(\Lambda)$ is dense in I, therefore there exists an interval $I_{\bar{\xi}} \subset I$ containing $\bar{\xi}$ such that δ_j can be extended continuously on $I_{\bar{\xi}}$. In addition, δ_1 satisfies

(3.11) $$|\delta_1(\bar{\xi})| = \left| e^{\pi i \bar{\eta}_0} + e^{\pi i \bar{\eta}_1} \right| < 2,$$

whenever $\bar{\xi} \in I \cap \Pi^{2^*}(\Lambda)$. Since δ_1 is continuous on $I_{\bar{\xi}}$, we can extract an interval $I' \subset I_{\bar{\xi}}$ containing $\bar{\xi}$ such that $|\delta_1(\xi)| < 2$ for all $\xi \in I'$.

Consequently, $I' \cap \Pi^{2^*}(\Lambda)$ is dense in I'. Now for $\xi \in I'$, there exists a sequence $\xi_n \in I' \cap \Pi^{2^*}(\Lambda)$ such that $\xi_n \to \xi$. Hence the corresponding image sequences $\eta_j^{(n)} \in \Sigma_{\xi_n} \subseteq [0, 2)$ will have convergent subsequences, say $\eta_j^{(n_k)}$ which converge to η_j; $j = 0, 1$. Since the set Λ is closed, $(\xi, \eta_j) \in \Lambda$ for $j = 0, 1$. Now, we only need to show that $\eta_0 \neq \eta_1$. If possible, suppose $\eta_0 = \eta_1$, then by the continuity of δ_1 on I', it follows that $|\delta_1(\xi_n)| \to |\delta_1(\xi)|$. However,

$$|\delta_1(\xi_{n_k})| = \left| e^{\pi i \eta_0^{(n_k)}} + e^{\pi i \eta_1^{(n_k)}} \right| \to 2.$$

That is, $|\delta_1(\xi)| = 2$, which contradicts the fact that $|\delta_1(\xi)| < 2$ for all $\xi \in I'$. Thus, we infer that $I' \subset \bigcup\limits_{j=2}^{4} \Pi^j(\Lambda)$. $\qquad \square$

Lemma 3.8. *Let I be an interval such that $I \cap \Pi^{3^*}(\Lambda)$ is dense in I. Then there exists an interval $I' \subset I$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$.*

Proof. Let $\bar{\xi} \in I \cap \Pi^{3^*}(\Lambda)$, then $e_j \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$; $j = 0, 1, 2$. For $p = 3$, Equation (3.10) yields

$$(e_0, e_1, e_2) = (-abc, (ab + bc + ca), -(a + b + c)),$$

where $(a, b, c) = (\chi_0, \chi_1, \chi_2)$. Hence e_j; $j = 0, 1, 2$ are constant multiples of the elementary symmetric polynomials. Now, we define a function ρ on $\Pi^3(\Lambda)$ by

$$\rho = \left(a^3(b - c) + b^3(c - a) + c^3(a - b) \right)^2.$$

Since ρ is a symmetric polynomial in a, b, c, by the fundamental theorem of symmetric polynomials, ρ can be expressed as a polynomial in e_j; $j = 0, 1, 2$. Moreover, $\rho(\bar{\xi}) \neq 0$. Hence it follows that $\rho \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$. By hypothesis, $I \cap \Pi^{3^*}(\Lambda)$ is dense in I, there exists an interval $I_{\bar{\xi}} \subset I$ containing $\bar{\xi}$ such that

ρ can be continuously extended on $I_{\bar{\xi}}$. Thus, by continuity of ρ on $I_{\bar{\xi}}$, there exists an interval $J \subset I_{\bar{\xi}}$ containing $\bar{\xi}$ such that $\rho(\xi) \neq 0$ for all $\xi \in J$.

Consequently, $J \cap \Pi^{3^*}(\Lambda)$ is dense in J and hence for $\xi \in J$, there exists a sequence $\xi_n \in J \cap \Pi^{3^*}(\Lambda)$ such that $\xi_n \to \xi$. Thus, the corresponding image sequences $\eta_j^{(n)} \in \Sigma_{\xi_n} \subseteq [0,2)$ will have convergent subsequences, say $\eta_j^{(n_k)}$ which converge to η_j; $j = 0, 1, 2$. Since the set Λ is closed, $(\xi, \eta_j) \in \Lambda$ for $j = 0, 1, 2$.

Next, we claim that all of η_j; $j = 0, 1, 2$ are distinct. On the contrary, suppose all are equal or any two of them are equal. Then by the continuity of ρ on J, it follows that $\rho(\xi) = 0$, which contradicts the fact that $\rho(\xi) \neq 0$ for all $\xi \in J$. Hence we infer that $J \subset \bigcup_{j=3}^{4} \Pi^j(\Lambda)$. Further, using the facts that $e_j \in L_{loc}^{\Pi^3(\Lambda),\bar{\xi}}$ and $J \cap \Pi^{3^*}(\Lambda)$ is dense in J, e_j can be extended continuously on an interval $I' \subset J$ containing $\bar{\xi}$ such that $e_j(\xi) \neq 0$ for all $\xi \in I'$. That is, if $\xi \in I' \cap \Pi^3(\Lambda)$, then $e_j \in L_{loc}^{\Pi^3(\Lambda),\xi}$ and hence $\xi \in \Pi^{3^*}(\Lambda)$. Thus, we conclude that $I' \subset \Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$. $\qquad\square$

Proof of Theorem 3.5. We first prove the sufficient part of Theorem 3.5. Suppose the set $\widetilde{\Pi_p}(\Lambda)$ is dense in $\mathbb{R}$. Then we show that (Γ, Λ) is a Heisenberg uniqueness pair. For $\hat{\mu}|_{\Lambda} = 0$, we claim that $\hat{f}_k|_{\widetilde{\Pi_p}(\Lambda)} = 0$, whenever $k \in F_o$. Since $\hat{f}_k$ is a continuous function which vanishes on a dense set $\widetilde{\Pi_p}(\Lambda)$, it follows that $\hat{f}_k \equiv 0$ for all $k \in F_o$. Thus, $\mu = 0$.

As the projection $\Pi(\Lambda)$ is decomposed into the four pieces, the proof of the above assertion will be carried out in the following four cases.

($\mathbf{S_1}$). If $\xi \in \Pi^4(\Lambda)$, then there exist at least four distinct $\eta_j \in \Sigma_\xi$ such that $\hat{\mu}(\xi, \eta_j) = 0$ for all $j \in F_o$. Hence $\hat{f}_k(\xi)$; $k \in F_o$ satisfy a homogeneous system of four equations. As $\xi \in \Pi^4(\Lambda)$, by using the property that $H_{p-2}(a_0, a_1, a_2) \neq H_{p-2}(a_0, a_1, a_3)$, we infer that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

($\mathbf{S_2}$). If $\xi \in \Pi^3(\Lambda)$, then there exist at least three distinct $\eta_j \in \Sigma_\xi$ which satisfy $\hat{\mu}(\xi, \eta_j) = 0$; $j = 0, 1, 2$. If $\hat{f}_3(\xi) = 0$, then we get $\hat{f}_k(\xi) = 0$ for $k = 0, 1, 2$. On the other hand if $\hat{f}_3(\xi) \neq 0$, then we can substitute

$$(3.12) \qquad \hat{f}_j(\xi) = \tau_j(\xi)\hat{f}_3(\xi),$$

where τ_j are defined on $\Pi^3(\Lambda)$ for $j = 0, 1, 2$. Hence $X_\xi = (\tau_0(\xi), \tau_1(\xi), \tau_2(\xi))$ will satisfy the system of equations $A_\xi^3 X_\xi = B_\xi^p$. By applying the Wiener lemma to Equations (3.12), we infer that $\tau_j \in L_{loc}^{\Pi^3(\Lambda),\xi}$; $j = 0, 1, 2$. That is, $\xi \in \Pi_{3^*}^p(\Lambda)$. Thus for $\xi \in \Pi^3(\Lambda) \smallsetminus \Pi_{3^*}^p(\Lambda)$, we conclude that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

($\mathbf{S_3}$). If $\xi \in \Pi^2(\Lambda)$, then there exist two distinct $\eta_j \in \Sigma_\xi$ for which $\hat{\mu}(\xi, \eta_j) = 0$, whenever $j = 0, 1$. That is,

$$(3.13) \qquad \hat{f}_0(\xi) + \chi_j(\xi)\hat{f}_1(\xi) + \chi_j^2(\xi)\hat{f}_2(\xi) + \chi_j^p(\xi)\hat{f}_3(\xi) = 0,$$

where $\chi_j(\xi) = e^{\pi i \eta_j}$; $j = 0, 1$. If $\hat{f}_3(\xi) \neq 0$, then by applying the Wiener lemma to Equations (3.13), it follows that $\xi \in \Pi_{2*}^p(\Lambda)$. That is, if $\xi \in \Pi^2(\Lambda) \smallsetminus \Pi_{2*}^p(\Lambda)$, then $\hat{f}_3(\xi) = 0$.

Further, if $\hat{f}_3(\xi) = 0$ and $\hat{f}_2(\xi) \neq 0$, then an application of the Wiener lemma to Equations (3.13), it follows that $\xi \in \Pi^{2^*}(\Lambda)$. By Lemma 3.3, $\xi \in \Pi_{2*}^p(\Lambda)$. Thus for $\xi \in \Pi^2(\Lambda) \smallsetminus \Pi_{2*}^p(\Lambda)$, we infer that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

($\mathbf{S_4}$). If $\xi \in \Pi^1(\Lambda)$, then there exists a unique $\eta_0 \in \Sigma_\xi$ for which $\hat{\mu}(\xi, \eta_0) = 0$. That is,

$$(3.14) \qquad \hat{f}_0(\xi) + \chi_0(\xi)\hat{f}_1(\xi) + \chi_0^2(\xi)\hat{f}_2(\xi) + \chi_0^p(\xi)\hat{f}_3(\xi) = 0,$$

where $\chi_0(\xi) = e^{\pi i \eta_0}$. If $\hat{f}_3(\xi) \neq 0$, then by applying the Wiener lemma to Equation (3.14), it implies that $\chi_0 \in P^{1,p}[L_{loc}^{\Pi^1(\Lambda), \xi}]$. That is, $\xi \in \Pi^{1^*}(\Lambda)$. Thus for $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, we have $\hat{f}_3(\xi) = 0$.

Further, if $\hat{f}_3(\xi) = 0$ and $\hat{f}_2(\xi) \neq 0$, then an application of the Wiener lemma to Equation (3.14), yields $\chi_0 \in P^{1,2}[L_{loc}^{\Pi^1(\Lambda), \xi}]$. By Lemma 3.2, it follows that $\xi \in \Pi^{1^*}(\Lambda)$. That is, if $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, then $\hat{f}_k(\xi) = 0$ for $k = 2, 3$.

Finally, if $\hat{f}_k(\xi) = 0$ for $k = 2, 3$ and $\hat{f}_1(\xi) \neq 0$, then by applying the Wiener lemma to Equation (3.14), we infer that $\chi_0 \in L_{loc}^{\Pi^1(\Lambda), \xi}$. By Lemma 3.2, it follows that $\xi \in \Pi^{1^*}(\Lambda)$. Thus for $\xi \in \Pi^1(\Lambda) \smallsetminus \Pi^{1^*}(\Lambda)$, we conclude that $\hat{f}_k(\xi) = 0$ for all $k \in F_o$.

Now, we prove the necessary part of Theorem 3.5. Suppose (Γ, Λ) is a Heisenberg uniqueness pair. Then we claim that the set $\widetilde{\Pi}(\Lambda)$ is dense in $\mathbb{R}$. We observe that this is possible if the dispensable sets $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$ interlace to each other, though these sets are disjoint among themselves.

If possible, suppose $\widetilde{\Pi}(\Lambda)$ is not dense in $\mathbb{R}$. Then there exists an open interval $I_o \subset \mathbb{R}$ such that $I_o \cap \widetilde{\Pi}(\Lambda)$ is empty. This in turn implies that

$$(3.15) \qquad \Pi(\Lambda) \cap I_o = \left(\bigcup_{j=1}^{3} \Pi^{j^*}(\Lambda) \right) \cap I_o.$$

Thus from (3.15), it follows that I_o intersects only the dispensable sets. Now, the remaining part of the proof of Theorem 3.5 is a consequence of the following two lemmas which provide the interlacing property of the dispensable sets $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$.

Lemma 3.9. *There does not exist any interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J$ is contained in $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$.*

Proof. On the contrary, suppose there exists an interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J \subset \Pi^{j^*}(\Lambda)$, for some $j \in \{1,2,3\}$. Since $\Pi^{j^*}(\Lambda)$; $j = 1,2,3$ are disjoint among themselves, there could be three possibilities.

(**a**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda)$, then $\chi_0 \in P^{1,p}[L_{loc}^{\Pi^1(\Lambda),\xi}]$. Hence there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$; $k = 0,1,2$ such that

$$\chi_0^p + \hat{\varphi}_2\chi_0^2 + \hat{\varphi}_1\chi_0 + \hat{\varphi}_0 = 0$$

on $I_\xi \cap \Pi^1(\Lambda)$. Now, consider a function $f_3 \in L^1(\mathbb{R})$ such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$. Let $f_k = f_3 * \varphi_k$; $k = 0,1,2$. Then we can construct a Borel measure μ which is supported on Γ such that

$$\hat{\mu}(\overline{\xi}, \overline{\eta}) = \hat{f}_0(\overline{\xi}) + \chi_0(\overline{\xi})\hat{f}_1(\overline{\xi}) + \chi_0^2(\overline{\xi})\hat{f}_2(\overline{\xi}) + \chi_0^p(\overline{\xi})\hat{f}_3(\overline{\xi}) = 0$$

for all $\overline{\xi} \in I_\xi \cap \Pi^{1^*}(\Lambda)$, where $\overline{\eta} \in \Sigma_{\overline{\xi}}$. Since (3.15) yields $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{1^*}(\Lambda)$, it implies that $\hat{\mu}|_\Lambda = 0$. However, μ is a non-zero measure which contradicts the fact that (Γ, Λ) is a HUP.

(**b**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{2^*}(\Lambda)$, then by Lemma 3.3, $\xi \in \Pi_{2*}^p(\Lambda)$. Hence there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$; $k = 0,1,2$ such that

$$\chi_j^p + \hat{\varphi}_2\chi_j^2 + \hat{\varphi}_1\chi_j + \hat{\varphi}_0 = 0$$

on $I_\xi \cap \Pi^2(\Lambda)$ for $j = 0,1$. Let $f_3 \in L^1(\mathbb{R})$ be such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$. Denote $f_k = f_3 * \varphi_k$; $k = 0,1,2$. Then we can construct a Borel measure μ that satisfies

$$\hat{\mu}(\bar{\xi}, \bar{\eta}_j) = \hat{f}_0(\bar{\xi}) + \chi_j(\bar{\xi})\hat{f}_1(\bar{\xi}) + \chi_j^2(\bar{\xi})\hat{f}_2(\bar{\xi}) + \chi_j^p(\bar{\xi})\hat{f}_3(\bar{\xi}) = 0$$

for all $\bar{\xi} \in I_\xi \cap \Pi^{2^*}(\Lambda)$ and $j = 0,1$. Since $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{2^*}(\Lambda)$, it follows that $\hat{\mu}|_\Lambda = 0$, though μ is a non-zero measure.

(**c**). If $\xi \in \Pi(\Lambda) \cap J \subset \Pi^{3^*}(\Lambda)$, then by Lemma 3.4, it follows that $\xi \in \Pi_{3*}^p(\Lambda)$. As $\tau_k \in L_{loc}^{\Pi^3(\Lambda),\xi}$; $k = 0,1,2$, there exists an interval $I_\xi \subset J$ containing ξ and $\varphi_k \in L^1(\mathbb{R})$ such that $\hat{\varphi}_k = \tau_k$ on $I_\xi \cap \Pi^3(\Lambda)$ for $k = 0,1,2$. Let $f_3 \in L^1(\mathbb{R})$ be such that $\hat{f}_3(\xi) \neq 0$ and supp $\hat{f}_3 \subset I_\xi$. Denote $f_k = f_3 * \varphi_k$; $k = 0,1,2$. Since $X_\xi = (\tau_0(\xi), \tau_1(\xi), \tau_2(\xi))$ satisfies $A_\xi^3 X_\xi = B_\xi^p$, we have

$$\tau_0 + \chi_j\tau_1 + \chi_j^2\tau_2 + \chi_j^p = 0$$

on $I_\xi \cap \Pi^3(\Lambda)$ for $j = 0,1,2$. Hence we can construct a Borel measure μ such that

$$\hat{\mu}(\bar{\xi}, \bar{\eta}_j) = \hat{f}_0(\bar{\xi}) + \chi_j(\bar{\xi})\hat{f}_1(\bar{\xi}) + \chi_j^2(\bar{\xi})\hat{f}_2(\bar{\xi}) + \chi_j^p(\bar{\xi})\hat{f}_3(\bar{\xi}) = 0$$

for all $\bar{\xi} \in I_\xi \cap \Pi^{3^*}(\Lambda)$ and $j = 0,1,2$. As $I_\xi \cap \Pi(\Lambda) = I_\xi \cap \Pi^{3^*}(\Lambda)$, we infer that $\hat{\mu}|_\Lambda = 0$, even though μ is a non-zero measure. $\square$

The next lemma is to deal with the situation that any interval $J \subset I_o$ can not contain only the points of any pair of dispensable sets.

Lemma 3.10. *There does not exist any interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J$ is contained in $\Pi^{j^*}(\Lambda) \cup \Pi^{k^*}(\Lambda) \ \forall \ j \neq k$ and $j, k \in \{1,2,3\}$.*

Proof. On the contrary, suppose there exists an interval $J \subset I_o$ such that $\Pi(\Lambda) \cap J \subset \Pi^{j^*}(\Lambda) \cup \Pi^{k^*}(\Lambda)$ for some $j \neq k$ and $j, k \in \{1, 2, 3\}$. Then we have the following three cases:

(a). If $\Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)$, then Equation (3.15) yields

$$(3.16) \qquad\qquad J \cap \Pi(\Lambda) = J \cap \left(\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda) \right).$$

We claim that $J \cap \Pi^{2^*}(\Lambda)$ is dense in J. If possible, suppose there exists an interval $I \subset J$ such that $\Pi^{2^*}(\Lambda) \cap I = \varnothing$. Then from (3.16), we get $I \cap \Pi(\Lambda) = I \cap \Pi^{1^*}(\Lambda) \subset \Pi^{1^*}(\Lambda)$ which contradicts Lemma 3.9. By Lemma 3.7, there exists an interval $I' \subset J$ such that $I' \subset \bigcup_{j=2}^{4} \Pi^j(\Lambda)$. This contradicts the assumption that I_o intersects only the dispensable sets.

(b). If $\Pi(\Lambda) \cap J \subset \Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)$, then $J \cap \Pi(\Lambda) = J \cap \left(\Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda) \right)$. As similar to the case **(a)**, $J \cap \Pi^{3^*}(\Lambda)$ is also dense in J. Hence by Lemma 3.8, there exists an interval $I' \subset J$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$. Thus in view of Lemma 3.9, we have arrived at a contradiction to the assumption that I_o intersects only the dispensable sets.

(c). If $\Pi(\Lambda) \cap J \subset \Pi^{2^*}(\Lambda) \cup \Pi^{3^*}(\Lambda)$, then $J \cap \Pi(\Lambda) = J \cap (\Pi^{2^*}(\Lambda) \cup \Pi^{3^*}(\Lambda))$. Hence it follows that $J \cap \Pi^{3^*}(\Lambda)$ is dense in J. By using Lemma 3.8, there exists an interval $I' \subset J$ such that I' is contained in $\Pi^{3^*}(\Lambda) \cup \Pi^4(\Lambda)$, which contradict the assumption that I_o intersects only the dispensable sets. $\square$

Finally, since $\Pi(\Lambda)$ is a dense subset of $\mathbb{R}$, in view of Lemmas 3.9 and 3.10, the only possibility that any interval $J \subset I_o$ would intersect all the dispensable sets $\Pi^{j^*}(\Lambda)$; $j = 1, 2, 3$. We claim that $\Pi^{2^*}(\Lambda) \cap I_o$ is dense in I_o. Otherwise, there exists an interval $I \subset I_o$ such that $\Pi^{2^*}(\Lambda) \cap I = \varnothing$. Then from (3.15), we get $I \cap \Pi(\Lambda) \subset \left(\Pi^{1^*}(\Lambda) \cup \Pi^{3^*}(\Lambda) \right)$ which contradicts Lemma 3.10. Hence by Lemma 3.7, there exists an interval $I' \subset I_o$ such that I' is contained in $\Pi^2(\Lambda) \cup \Pi^3(\Lambda) \cup \Pi^4(\Lambda)$ which contradicts the assumption that I_o intersects only the dispensable sets.

Concluding remarks:

(a). We observe a phenomenon of interlacing of three totally disconnected disjoint dispensable sets $\Pi^{(3-j)^*}(\Lambda) : j = 0, 1, 2$ which are essentially derived from zero sets of four trigonometric polynomials.

(b). If the measure in question is supported on an arbitrary number of parallel lines, then the size of the dispensable sets would be larger. Indeed, the method used for the proof of Theorem 3.5 would be highly implicit for a large number of parallel lines. Since the dispensable sets are totally disconnected, it would be an interesting question to analyze Heisenberg uniqueness pairs corresponding to the finite number of parallel lines in terms of Hausdorff dimension of the dispensable sets.

(c). If we consider countably many parallel lines, then whether the projection $\Pi(\Lambda)$ would be still relevant after deleting the countably many dispensable sets, seems to be a reasonable question. We leave these questions open for the time being.

(d). For $p = 3$, in Lemma 3.8 we have used the fact that any symmetric polynomial in a, b, c can be expressed as a polynomial in τ_j; $j = 0, 1, 2$. This enables us to define a function $\rho \in L_{loc}^{\Pi^3(\Lambda), \bar{\xi}}$, which is crucial in the proof of Lemma 3.8. However, for $p \geq 4$, the functions τ_j; $j = 0, 1, 2$ appeared in Equations (3.10) are away from the elementary symmetric polynomials. If we could identify the space of symmetric polynomials generated by τ_j; $j = 0, 1, 2$, then we can think to modify the Lemma 3.8 in terms of $\Pi_{3*}^p(\Lambda)$ that would help in minimizing the size of the set $\widetilde{\Pi}(\Lambda)$. Hence a characterization of Λ for four lines problem might be obtained that would be closed to three lines result. However, an exact analogue of three lines result for a large number of lines is still open.

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References

[1] M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure,* J. Math. Anal. Appl. 106 (1985), no.1, 180-183.

[2] D. B. Babot, *Heisenberg uniqueness pairs in the plane, Three parallel lines,* Proc. Amer. Math. Soc. 141 (2013), no. 11, 3899-3904.

[3] S. C. Bagchi and A. Sitaram, *Determining sets for measures on $\mathbb{R}^n$*, Illinois J. Math. 26 (1982), no. 3, 419-422.

[4] W. F. Donoghue, Jr., *Distribution and Fourier transforms,* Academic Press, New York, 1969.

[5] L. Flatto, D. J. Newman, and H. S. Shapiro, *The level curves of harmonic polynomials,* Trans. Amer. Math. Soc. 123 (1966), 425-436.

[6] V. Havin, and B. Jricke, *The Uncertainty Principle in Harmonic Analysis,* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 28. Springer-Verlag, Berlin, 1994.

[7] H. Hedenmalm, A. Montes-Rodrguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation,* Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.

[8] P. Jaming and K. Kellay, *A dynamical system approach to Heisenberg uniqueness pairs,* J. Anal. Math. (appearing) arXiv:1312.6236, July, 2014.

[9] J.P. Kahane, *Sries de Fourier absolument convergentes,* Band 50 Springer-Verlag, Berlin-New York 1970.

[10] N. Lev, *Uniqueness theorem for Fourier transform. Bull. Sc. Math.,* 135(2011), 134-140.

[11] I.G. Macdonald, *Symmetric Functions and Hall Polynomials,* Second ed., Oxford University Press, New York, 1995.

[12] E. K. Narayanan and P. K. Ratnakumar, *Benedick's theorem for the Heisenberg group,* Proc. Amer. Math. Soc. 138 (2010), no. 6, 2135-2140.

[13] V. Pati and A. Sitaram, *Some questions on integral geometry on Riemannian manifolds,* Ergodic theory and harmonic analysis (Mumbai, 1999). Sankhya Ser. A 62 (2000), no. 3, 419-424.

[14] R. Rawat, A. Sitaram, *Injectivity sets for spherical means on $\mathbb{R}^n$ and on symmetric spaces,* Jour. Four. Anal. Appl., 6 (2000), 343-348.

[15] F. J. Price and A. Sitaram, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups,* J. Funct. Anal. 79 (1988), no. 1, 166-182.

[16] P. Sjolin, *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin,* Bull. Sc. Math., 135(2011), 125-133.

[17] P. Sjolin, *Heisenberg uniqueness pairs for the parabola,* Jour. Four. Anal. Appl., 19(2013), 410-416.

[18] A. Sitaram, M. Sundari and S. Thangavelu, *Uncertainty principles on certain Lie groups,* Proc. Indian Acad. Sci. Math. Sci. 105 (1995), no. 2, 135-151.

[19] F. J. Gonzalez Vieli, *A uniqueness result for the Fourier transform of measures on the sphere,* Bull. Aust. Math. Soc. 86(2012), 78-82.

[20] F. J. Gonzalez Vieli, *A uniqueness result for the Fourier transform of measures on the paraboloid,* Matematicki Vesnik, Vol. 67(2015), No. 1, 52-55.

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